

# Non-linear vorticity-density coupling in Lagrangian dynamics

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We present a general Lagrangian formalism that allows the treatment of vorticity. We give solutions for the rotational perturbations up to the third-order in a flat background universe. We show how the primordial vorticity affects the evolution of the density fluctuation in high-density regions.

## §1. Introduction

The study of the galaxy formation based on gravitational instability has not revealed the origin of the vorticity of galaxies. The possibility that galactic spins are the relics of primordial vorticity has not been taken into account. Because the standard linear Eulerian theory (see, for example, Peebles' book<sup>1)</sup> for the detail of the Eulerian theory) shows that the vorticity decays at late time and does not couple to the density enhancement. For this reason, the methods that generate the vorticity posteriorly are investigated. Doroshkevich<sup>2)</sup> explained the acquisition of the vorticity by the formation of shock fronts in the pancake. The explanation by tidal torques was made by Hoyle.<sup>3)</sup>

However, can we actually ignore the effects of the primordial vorticity in the galaxy formation process? To answer this question, we must reconsider the validity of the Eulerian theory. In the Eulerian treatment, density fluctuation is perturbative quantity, so it must be small. Therefore, the Eulerian approach is limited to the description of weak density fluctuation. For this reason, to discuss the behavior of the vorticity in the high-density regions, we must prepare the theory beyond the Eulerian theory. The Lagrangian theory improves such shortcomings of the Eulerian theory. The most advantageous point of the Lagrangian theory is that the density fluctuation is non-perturbative quantity. Therefore, it is unnecessary that the density fluctuation is expanded in a perturbation series and that we impose the condition of the smallness of the density fluctuation. By using the property of the Lagrangian theory, we can investigate the behavior of the primordial vorticity. Buchert<sup>4)</sup> solved the first-order Lagrangian perturbative equations and showed that the primordial vorticity was amplified in proportion to the enhancement of the density fluctuation by deriving the relation between the vorticity and the density fluctuation. Barrow and Saich<sup>5)</sup> solved the vorticity equation on the inhomogeneous background and pointed out that the growing directions of the vorticity were lying in the plane of the pancake. These analyses based on the Lagrangian theory shows that the primordial vorticity is not negligible in the high-density regions such as the pancake.

In this paper, for the purpose of preparing a tool to treat the behavior of the vorticity in the high-density regions, we extend the previous works and give solutions including the vorticity for the Lagrangian perturbative equations up to the third-

order. Furthermore, we obtain the relation between the vorticity and the density fluctuation. The reason we proceed to the calculation up to the third-order is that the third-order solutions will cover the main effects since the density fluctuation is described by the determinant of a  $3 \times 3$  matrix. In the Lagrangian approach, we assume that background is Friedmann-Lemaître-Robertson-Walker (FLRW) universe with no cosmological constant and curvature. There is no definitive observational evidence that our universe is actually flat. We apply this assumption for simplifying calculations.

This paper is organized as follows. In §2, we summarize the basic formulae of the Lagrangian theory. In §3, we present the perturbative approach to the Lagrangian equations including the treatment of the vorticity. In §4, we summarize the main results of this paper.

## §2. Basic equations in Lagrangian form

In this section, we give a summary of the Lagrangian theory. We particularly focus on the treatment of the vorticity in the Lagrangian theory. The detail of the Lagrangian theory is found, for example, in Ref. 6).

### 2.1. Density fluctuation in Lagrangian form

In this subsection, we focus on the expression of the density fluctuation in Lagrangian form.

In the Eulerian perturbation theory, the density fluctuation  $\delta$  is a perturbative quantity. On the other hand, in the Lagrangian form, the density fluctuation is given in the formally exact form as follows:

$$\delta = J^{-1} - 1, \quad J := \det \left( \frac{\partial x_i}{\partial q_j} \right), \quad (2.1)$$

where  $\mathbf{x}$  are expanding coordinates in the Eulerian space and  $\mathbf{q}$  are the Lagrangian coordinates which are defined by the initial Eulerian position, namely,  $\mathbf{q} := \mathbf{x}(t = t_0)$ . (The physical distance is obtained according to the law  $\mathbf{r} = a(t)\mathbf{x}$ ,  $a(t)$  being the scale factor.) The determinant  $J$  of the Jacobian of the transformation from  $\mathbf{x}$  to  $\mathbf{q}$  is basic quantity to represent the density fluctuation. Since  $J$  is defined by the Lagrangian partial derivative, we must evaluate  $\mathbf{x}$  in the Lagrangian space:  $\mathbf{x} = \mathbf{x}(\mathbf{q}, t)$ . Therefore, it is necessary to clarify the relation between the Eulerian coordinates and the Lagrangian ones. The displacement vector is the quantity that relates  $\mathbf{x}$  with  $\mathbf{q}$ . This quantity represents time evolution from the initial position  $\mathbf{q}$ . By using the displacement vector  $\mathbf{A}(\mathbf{q}, t)$ , we can write the relation between  $\mathbf{x}$  and  $\mathbf{q}$  as  $\mathbf{x} = \mathbf{q} + \mathbf{A}(\mathbf{q}, t)$ . If we obtain the concrete form of  $\mathbf{A}$ , the density fluctuation is calculated from Eq. (2.1). Since Eq. (2.1) is derived from mass conservation between the Eulerian and the Lagrangian space, this equation is hold irrespective of the existence of the vorticity.

Now we shall consider how the effects of the vorticity on the density fluctuation are reflected. A conventional way to investigate the effects of the vorticity is that we split  $\mathbf{A}$  into the longitudinal mode and the transverse one with respect to the

the Eulerian coordinates and then focus on the behavior of the transverse mode. However, since  $\mathbf{A}$  is evaluated in the Lagrangian coordinates, we shall perform this decomposition with respect to the Lagrangian coordinates:

$$\mathbf{A} = \mathbf{A}^{\parallel} + \mathbf{A}^{\perp}, \quad \nabla_q \times \mathbf{A}^{\parallel} = 0, \quad \nabla_q \cdot \mathbf{A}^{\perp} = 0, \quad (2.2)$$

where the superscript  $\parallel$  and  $\perp$  denote the longitudinal and the transverse mode with respect to the Lagrangian coordinates, respectively. Although the quantity  $\mathbf{A}^{\perp}$  does not correspond with the vorticity, we shall express the density fluctuation by using  $\mathbf{A}^{\parallel}$  and  $\mathbf{A}^{\perp}$ . The method to express the vorticity in the Lagrangian space will be presented in next subsection. From the definition of  $J$  in Eq. (2.1), we obtain

$$\begin{aligned} J = 1 + \nabla_q \cdot \mathbf{A}^{\parallel} &+ \frac{1}{2} \left[ \left( \nabla_q \cdot \mathbf{A}^{\parallel} \right)^2 - A_{i,j}^{\parallel} A_{j,i}^{\parallel} \right] - A_{i,j}^{\parallel} A_{j,i}^{\perp} - \frac{1}{2} A_{i,j}^{\perp} A_{j,i}^{\perp} \\ &+ \det \left( A_{i,j}^{\parallel} + A_{i,j}^{\perp} \right), \end{aligned} \quad (2.3)$$

where

$$\begin{aligned} \det(A_{i,j}^{\parallel} + A_{i,j}^{\perp}) = &\frac{1}{3} A_{i,j}^{\parallel} A_{j,k}^{\parallel} A_{k,i}^{\parallel} - \frac{1}{2} A_{i,j}^{\parallel} A_{j,i}^{\parallel} A_{l,l}^{\parallel} + \frac{1}{6} \left( A_{i,i}^{\parallel} \right)^3 \\ &+ A_{i,j}^{\parallel} A_{j,k}^{\perp} A_{k,i}^{\parallel} - A_{i,j}^{\parallel} A_{j,i}^{\perp} A_{l,l}^{\parallel} \\ &+ A_{i,j}^{\perp} A_{j,k}^{\perp} A_{k,i}^{\parallel} - \frac{1}{2} A_{i,j}^{\perp} A_{j,i}^{\perp} A_{l,l}^{\parallel} \\ &+ \frac{1}{3} A_{i,j}^{\perp} A_{j,k}^{\perp} A_{k,i}^{\perp} \end{aligned} \quad (2.4)$$

and the comma denotes the partial derivative with respect to the Lagrangian coordinates. By considering the transverse mode  $\mathbf{A}^{\perp}$ , we can investigate how the vorticity contributes to the density enhancement.

In next subsection, we derive the Lagrangian equations for  $\mathbf{A}^{\parallel}$  and  $\mathbf{A}^{\perp}$  which are necessary for the calculation of the density fluctuation.

## 2.2. Equations for longitudinal and transverse mode in Lagrangian coordinates

As mentioned in previous subsection, we derive the Lagrangian equations for  $\mathbf{A}^{\parallel}$  and  $\mathbf{A}^{\perp}$  here. The derivation of the equations is based on the methods by Buchert and Götz,<sup>7)</sup> and Kasai<sup>8)</sup>. Furthermore, we also present the expression of the vorticity in the Lagrangian space.

First, we shall derive the Lagrangian equations. We denote the density of dust, the background density, the peculiar velocity and the peculiar gravitational acceleration by  $\rho$ ,  $\rho_b$ ,  $\mathbf{v}$  and  $\mathbf{g}$ , respectively. The peculiar velocity given by the form  $\mathbf{v} = a\dot{\mathbf{x}}$  represents the deviation from the uniform Hubble flow. Since we assume the FLRW universe, the scale factor  $a$  is proportional to  $t^{2/3}$ . We consider that  $a = t^{2/3}$ . The basic equations in Newtonian cosmology that these quantities obey are

$$\frac{\partial \rho}{\partial t} + 3\frac{\dot{a}}{a}\rho + \frac{1}{a}\nabla_x \cdot (\rho \mathbf{v}) = 0, \quad (2.5a)$$

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{\dot{a}}{a} \mathbf{v} + \frac{1}{a} (\mathbf{v} \cdot \nabla_x) \mathbf{v} = \mathbf{g}, \quad (2.5b)$$

$$\nabla_x \times \mathbf{g} = 0, \quad (2.5c)$$

$$\nabla_x \cdot \mathbf{g} = -4\pi G a (\rho - \rho_b), \quad (2.5d)$$

where  $\nabla_x$  is the Eulerian nabla operator. We obtain from Eqs. (2.5a) and (2.5d)

$$\nabla_x \cdot \left( \frac{\partial \mathbf{g}}{\partial t} + 2 \frac{\dot{a}}{a} \mathbf{g} - 4\pi G \rho \mathbf{v} \right) = 0, \quad (2.6a)$$

and from Eqs. (2.5b) and (2.5c)

$$\nabla_x \times \left( \frac{\partial \mathbf{v}}{\partial t} + \frac{\dot{a}}{a} \mathbf{v} + \frac{1}{a} (\mathbf{v} \cdot \nabla_x) \mathbf{v} \right) = 0. \quad (2.6b)$$

Eqs. (2.6a) and (2.6b) are the equations for the longitudinal and the transverse mode with respect to the Eulerian coordinates, respectively. We rewrite the system (2.6) to the Lagrangian equations by transforming the Eulerian partial derivative into the Lagrangian one

$$\begin{aligned} \frac{\partial}{\partial x_i} &= \frac{\partial q_j}{\partial x_i} \frac{\partial}{\partial q_j} = \left( \delta_{ji} - \frac{\partial A_j}{\partial x_i} \right) \frac{\partial}{\partial q_j} \\ &:= (\delta_{ji} + B_{ji}) \frac{\partial}{\partial q_j} \end{aligned} \quad (2.7)$$

and introducing the Lagrangian time derivative

$$\frac{d}{dt} := \frac{\partial}{\partial t} + \frac{1}{a} \mathbf{v} \cdot \nabla_x. \quad (2.8)$$

The results are

$$\begin{aligned} &\frac{d}{dt} \left[ a^3 \left\{ \mathcal{D}(\nabla_q \cdot \mathbf{A}^{\parallel}) - 4\pi G \rho_b \nabla_q \cdot \mathbf{A}^{\parallel} \right\} \right] \\ &= -B_{ij} \left[ \frac{d}{dt} \left\{ a^3 (\mathcal{D}(A_{j,i}) - 4\pi G \rho_b A_{j,i}) \right\} \right] + a^3 \nabla_x \cdot \left[ (\dot{\mathbf{A}} \cdot \nabla_x) \mathcal{D}(\mathbf{A}) - (\nabla_x \cdot \mathcal{D}(\mathbf{A})) \dot{\mathbf{A}} \right], \end{aligned} \quad (2.9a)$$

$$\mathcal{D}(\epsilon_{ijk} A_{k,j}^{\perp}) = -\epsilon_{ijk} B_{lj} \mathcal{D}(A_{k,l}), \quad (2.9b)$$

where  $\mathcal{D}$  is the differential operator satisfying

$$\mathcal{D}(F) := \left( \frac{d^2 F}{dt^2} + 2 \frac{\dot{a}}{a} \frac{dF}{dt} \right), \quad (2.10)$$

$\epsilon_{ijk}$  is Levi-Civita asymmetric tensor,  $\epsilon_{123} := 1$ , and  $\nabla_q$  is the Lagrangian nabla operator. The detail of the second term in right-hand side of (2.9a) is in Appendix A. Eqs. (2.9a) and (2.9b) are the equations for the longitudinal and transverse mode with respect to the Lagrangian coordinates, respectively. The system (2.9) is the Lagrangian equations which are necessary for obtaining the density fluctuation.

Next, we shall consider the expression of the vorticity in the Lagrangian space. Here, we define the vorticity as  $\mathbf{w} := (1/a)\nabla_x \times \mathbf{v}$ . By transforming  $\mathbf{x}$  into  $\mathbf{q}$ , we obtain

$$w_i = \frac{1}{a}[\nabla_q \times \mathbf{v}]_i + \frac{1}{a}\epsilon_{ijk}B_{lj}v_{k,l}. \quad (2.11)$$

However, there exists another expression of the vorticity. In the Lagrangian representation, the equation for the transverse mode with respect to the Eulerian space (2.6b) is solved exactly as (see Ref. 4))

$$\mathbf{w} = \frac{1}{a^2}(1 + \delta)(\mathbf{w}(\mathbf{q}, t_0) \cdot \nabla_q)\mathbf{x}(\mathbf{q}, t), \quad (2.12)$$

where  $\mathbf{w}(\mathbf{q}, t_0)$  is the primordial vorticity. Eq. (2.12) represents that the vorticity couples to the enhancement of the density fluctuation. This fact is not respected in the Eulerian theory. Naturally, these two expressions of the vorticity (2.11) and (2.12) are equivalent. We stress that the Lagrangian method enables not only to calculate the density fluctuation straightforward but to clarify vorticity-density relation.

In next section, we will solve the Lagrangian equations by perturbative method and obtain the concrete expression of the density fluctuation and the vorticity.

### §3. Lagrangian perturbative approximation

In this section, we solve the Lagrangian equations (2.9a) and (2.9b) by perturbative method up to the third-order and obtain the determinant  $J$ . By calculating  $J$ , we obtain the concrete expression of the density fluctuation and the vorticity from Eqs. (2.1) and (2.12), respectively. Our treatment is different from the standard Lagrangian perturbation theory in considering the vorticity. Naturally, when the irrotational condition is assumed, our results are consistent with the standard Lagrangian perturbation theory. The reason the perturbative approach is introduced is that the right-hand sides of the system (2.9) are non-linear. However, since the density fluctuation itself has no restriction, Lagrangian perturbative approximation enables to describe the high-density regions. (The restriction by the Lagrangian perturbation approach is that we require the smallness of the deviation of the particle trajectory from the uniform Hubble flow.) In the Lagrangian perturbation approach, the perturbative quantity is the displacement vector:

$$\mathbf{x}(\mathbf{q}, t) = \mathbf{q} + \mathbf{A}_{(1)}(\mathbf{q}, t) + \mathbf{A}_{(2)}(\mathbf{q}, t) + \mathbf{A}_{(3)}(\mathbf{q}, t) + \cdots, \quad (3.1)$$

where  $\mathbf{A}_{(1)}(\mathbf{q}, t)$  corresponds to the first-order approximation,  $\mathbf{A}_{(2)}(\mathbf{q}, t)$  to the second-order approximation, and so on. We derive the perturbative equations for  $\mathbf{A}_{(1)}$ ,  $\mathbf{A}_{(2)}$  and  $\mathbf{A}_{(3)}$  and then solve these equations. Hereafter, we call the longitudinal (transverse) mode with respect to the Lagrangian coordinates simply “the longitudinal (transverse) mode”.

#### 3.1. First-order solutions

The first-order solutions for the longitudinal and the transverse mode are obtained by Buchert.<sup>4)</sup>

First, we consider the longitudinal mode. The equation for the first-order longitudinal mode is

$$\frac{d}{dt} \left[ a^3 \left\{ \mathcal{D}(\mathbf{A}_{(1)}^{\parallel}) - 4\pi G \rho_b \mathbf{A}_{(1)}^{\parallel} \right\} \right] = \nabla_q \times \mathbf{K}, \quad (3.2)$$

where  $\mathbf{K}$  is an arbitrary vector potential. The existence of the quantity  $\mathbf{K}$  means that the decomposition such as  $\mathbf{A} = \mathbf{A}^{\parallel} + \mathbf{A}^{\perp}$  is not unique; we may always add to  $\mathbf{A}^{\parallel}$  a rotation-free solution of  $\nabla_q \cdot \mathbf{A}^{\parallel} = 0$ . We can impose additional condition for eliminating this freedom. So we shall assume  $\nabla_q \cdot \mathbf{K} = 0$ . Integrating Eq. (3.2), we obtain

$$\mathcal{D}(\mathbf{A}_{(1)}^{\parallel}) - 4\pi G \rho_b \mathbf{A}_{(1)}^{\parallel} = \frac{\mathbf{C}}{a^3}, \quad \frac{d\mathbf{C}}{dt} = 0. \quad (3.3)$$

By choosing  $\mathbf{C} = 0$ , we obtain

$$\mathbf{A}_{(1)}^{\parallel} = t^{2/3} \mathbf{X}_{(1)}^{\parallel}(\mathbf{q}) + t^{-1} \mathbf{X}_{(1d)}^{\parallel}(\mathbf{q}). \quad (3.4)$$

(The Lagrangian perturbative solutions at any order separate with respect to  $\mathbf{q}$  and  $t$ . See Ref. 6.) The terms proportional to  $t^{2/3}$  and  $t^{-1}$  are called the growing mode and the decaying mode, respectively.

Next, we consider the transverse mode. The equation for the first-order transverse mode is

$$\mathcal{D}(\mathbf{A}_{(1)}^{\perp}) = \nabla_q \varphi, \quad (3.5)$$

where  $\varphi$  is an arbitrary scalar potential. We assume  $\nabla_q \varphi = 0$  for eliminating the freedom. The solution is

$$\mathbf{A}_{(1)}^{\perp} = t^{-1/3} \mathbf{X}_{(1)}^{\perp}(\mathbf{q}). \quad (3.6)$$

We shall call this solution the rotational mode.

### 3.2. Second-order solutions

The general irrotational second-order solutions are obtained by Buchert and Ehlers<sup>9)</sup> in the case of density parameter  $\Omega = 1$ . The leading terms of the irrotational second-order solution in the case of arbitrary density parameter are derived by Bouchet et al..<sup>10)</sup> In order to avoid notational complexity, we neglect the decaying mode  $\mathbf{X}_{(1d)}^{\parallel}$  hereafter.

First, we consider the longitudinal mode. The equation for the second-order longitudinal mode is

$$\begin{aligned} & \frac{d}{dt} \left[ a^3 \left\{ \mathcal{D}(\nabla_q \cdot \mathbf{A}_{(2)}^{\parallel}) - 4\pi G \rho_b \nabla_q \cdot \mathbf{A}_{(2)}^{\parallel} \right\} \right] \\ &= -B_{(1)ij} \left[ \frac{d}{dt} \left\{ a^3 \left( \mathcal{D}(A_{(1)j,i}) - 4\pi G \rho_b A_{(1)j,i} \right) \right\} \right] \\ &+ a^3 \left[ \dot{A}_{(1)i,j} \mathcal{D}(A_{(1)j,i}^{\parallel}) - \dot{A}_{(1)i,i} \mathcal{D}(A_{(1)j,j}^{\parallel}) \right]. \end{aligned} \quad (3.7)$$

The particular solution is

$$\mathbf{A}_{(2)}^{\parallel} = t^{4/3} \mathbf{X}_{(2a)}^{\parallel}(\mathbf{q}) + t^{-2/3} \mathbf{X}_{(2b)}^{\parallel}(\mathbf{q}), \quad (3.8)$$

where

$$\nabla_q \cdot \mathbf{X}_{(2a)}^{\parallel} = \frac{3}{14} \left[ X_{(1)i,j}^{\parallel} X_{(1)j,i}^{\parallel} - \left( \nabla_q \cdot \mathbf{X}_{(1)}^{\parallel} \right)^2 \right], \quad (3.9)$$

$$\nabla_q \cdot \mathbf{X}_{(2b)}^{\parallel} = \frac{3}{4} X_{(1)i,j}^{\perp} X_{(1)j,i}^{\perp}. \quad (3.10)$$

Although the solution for Eq. (3.7) consists of the homogeneous and the particular solution, we omit the homogeneous solution. We focus on the behavior of the first-order quantities, namely, the growing mode  $\mathbf{X}_{(1)}^{\parallel}$  and the rotational mode  $\mathbf{X}_{(1)}^{\perp}$ , in higher-order. Since the homogeneous equation is the same form as the first-order equation, the behavior of the homogeneous solutions is investigated easily. Now the solution (3.8) consists of two parts. One is  $\mathbf{X}_{(2a)}^{\parallel}$  described by using two growing modes. Another is  $\mathbf{X}_{(2b)}^{\parallel}$  described by using two rotational modes. In the case where the rotational mode is considered, the solution  $\mathbf{A}_{(2)}^{\parallel}$  includes the  $\mathbf{X}_{(1)}^{\perp}$ , although  $\mathbf{A}_{(2)}^{\parallel}$  describes the longitudinal mode. Furthermore, there is no cross term consisting of  $\mathbf{X}_{(1)}^{\parallel}$  and  $\mathbf{X}_{(1)}^{\perp}$ . This cross term is included the transverse mode as mentioned below.

Next, we consider the transverse mode. The equation for the second-order transverse mode is

$$\mathcal{D}(\epsilon_{ijk} A_{(2)k,j}^{\perp}) = -\epsilon_{ijk} B_{(1)lj} \mathcal{D}(A_{(1)k,l}). \quad (3.11)$$

In the same sense as that of the longitudinal mode, we shall obtain the particular solution for Eq. (3.11). The particular solution is

$$\mathbf{A}_{(2)}^{\perp} = t^{1/3} \mathbf{X}_{(2)}^{\perp}, \quad (3.12)$$

where

$$\left[ \nabla_q \times \mathbf{X}_{(2)}^{\perp} \right]_i = -3\epsilon_{ijk} X_{(1)j,l}^{\parallel} X_{(1)l,k}^{\perp}. \quad (3.13)$$

The second-order transverse mode is induced by the coupling of the rotational mode to the growing mode. Note that this coupling term is growing. On the other hand, the cross term of two  $\mathbf{X}_{(1)}^{\parallel}$  or two  $\mathbf{X}_{(1)}^{\perp}$  does not exist in Eq. (3.12). Therefore, if the rotational mode is zero, the second-order transverse mode vanishes. Actually, we must consider the fact that the decaying mode couples to  $\mathbf{X}_{(1)}^{\parallel}$  or  $\mathbf{X}_{(1)}^{\perp}$ . However, the coupling of the decaying mode is different from the coupling of  $\mathbf{X}_{(1)}^{\perp}$  to the  $\mathbf{X}_{(1)}^{\parallel}$  in decaying at late time. Although both the decaying mode and the rotational mode decay, the former is more effective when they couple to the growing mode in higher-order.

### 3.3. Third-order solutions

The irrotational third-order solution is obtained, for example, by Buchert<sup>11)</sup>, Bouchet et al.,<sup>12)</sup> and Catelan<sup>13)</sup>.

First, we consider the longitudinal mode. The equation for the third-order longitudinal mode is

$$\frac{d}{dt} \left[ a^3 \left\{ \mathcal{D}(\nabla_q \cdot \mathbf{A}_{(3)}^{\parallel}) - 4\pi G \rho_b \nabla_q \cdot \mathbf{A}_{(3)}^{\parallel} \right\} \right]$$

$$\begin{aligned}
&= -B_{(1)ij} \frac{d}{dt} \left[ a^3 \left\{ \mathcal{D}(A_{(2)j,i}) - 4\pi G \rho_b A_{(2)j,i} \right\} \right] \\
&\quad - B_{(2)ij} \frac{d}{dt} \left[ a^3 \left\{ \mathcal{D}(A_{(1)j,i}) - 4\pi G \rho_b A_{(1)j,i} \right\} \right] \\
&\quad + a^3 \left[ \dot{A}_{(1)i,j} \mathcal{D}(A_{(2)j,i}) + \dot{A}_{(2)i,j} \mathcal{D}(A_{(1)j,i}) - \dot{A}_{(1)i,i}^{\parallel} \mathcal{D}(A_{(2)j,j}^{\parallel}) - \dot{A}_{(2)i,i}^{\parallel} \mathcal{D}(A_{(1)j,j}^{\parallel}) \right] \\
&\quad + a^3 \left[ B_{(1)ij} \dot{A}_{(1)j,k} \mathcal{D}(A_{(1)k,i}) + B_{(1)ij} \mathcal{D}(A_{(1)j,k}) \dot{A}_{(1)k,i} \right. \\
&\quad \quad \left. - B_{(1)ij} \mathcal{D}(A_{(1)j,i}) \dot{A}_{(1)k,k}^{\parallel} - B_{(1)ij} \dot{A}_{(1)j,i} \mathcal{D}(A_{(1)k,k}^{\parallel}) \right]. \quad (3.14)
\end{aligned}$$

The particular solution is

$$\mathbf{A}_{(3)}^{\parallel} = t^2 \mathbf{X}_{(3a)}^{\parallel} + t \mathbf{X}_{(3b)}^{\parallel} + \frac{2}{75} t^{-1} (3 + 5 \ln t) \mathbf{X}_{(3c)}^{\parallel}, \quad (3.15)$$

where

$$\nabla_q \cdot \mathbf{X}_{(3a)}^{\parallel} = \frac{5}{9} \left[ X_{(1)i,j}^{\parallel} X_{(2a)j,i}^{\parallel} - X_{(1)i,i}^{\parallel} X_{(2a)j,j}^{\parallel} \right] - \frac{1}{3} \det(X_{(1)i,j}^{\parallel}), \quad (3.16)$$

$$\begin{aligned}
\nabla_q \cdot \mathbf{X}_{(3b)}^{\parallel} &= \frac{1}{3} X_{(1)i,j}^{\parallel} X_{(2)j,i}^{\perp} + \frac{7}{3} X_{(2a)i,j}^{\parallel} X_{(1)j,i}^{\perp} \\
&\quad - \left[ X_{(1)i,j}^{\parallel} X_{(1)j,k}^{\parallel} X_{(1)k,i}^{\perp} - X_{(1)i,j}^{\parallel} X_{(1)j,i}^{\perp} X_{(1)k,k}^{\parallel} \right], \quad (3.17)
\end{aligned}$$

$$\nabla_q \cdot \mathbf{X}_{(3c)}^{\parallel} = 2 X_{(2b)i,j}^{\parallel} X_{(1)j,i}^{\perp} - X_{(1)i,j}^{\perp} X_{(1)j,k}^{\perp} X_{(1)k,i}^{\perp}. \quad (3.18)$$

The third-order solution (3.15) does not contain the term consisting of two  $\mathbf{X}_{(1)}^{\perp}$  and one  $\mathbf{X}_{(1)}^{\parallel}$ . Furthermore, note that the third-order solution includes the term proportional to  $t^{-1} \ln t$ , although the solutions up to the second-order are described in the form of simple power of  $t$ . Since this term consists of three  $\mathbf{X}_{(1)}^{\perp}$ , the term is anticipated proportional to  $t^{-1}$ , namely,  $(t^{-1/3})^3$ . However, the solution proportional to  $t^{-1}$  is same form as the first-order decaying solution, that is, the homogeneous solution of the Lagrangian equation. Therefore, the term proportional to  $t^{-1} \ln t$  is necessary.

Next, we consider the transverse mode. The equation for the third-order transverse mode is

$$\mathcal{D}(\epsilon_{ijk} A_{(3)k,j}^{\perp}) = -\epsilon_{ijk} \left[ B_{(1)lj} \mathcal{D}(A_{(2)k,l}) + B_{(2)lj} \mathcal{D}(A_{(1)k,l}) \right]. \quad (3.19)$$

The particular solution is

$$\mathbf{X}_{(3)}^{\perp} = t^2 \mathbf{X}_{(3a)}^{\perp} + t \mathbf{X}_{(3b)}^{\perp} + \ln t \mathbf{X}_{(3c)}^{\perp} + t^{-1} \mathbf{X}_{(3d)}^{\perp}, \quad (3.20)$$

where

$$\epsilon_{ijk} X_{(3a)k,j}^{\perp} = \frac{1}{3} \epsilon_{ijk} X_{(1)l,j}^{\parallel} X_{(2a)k,l}^{\parallel}, \quad (3.21)$$

$$\epsilon_{ijk} X_{(3b)k,j}^{\perp} = -\frac{1}{3} \epsilon_{ijk} \left[ X_{(1)l,j}^{\parallel} X_{(2)l,k}^{\perp} + 5 X_{(2a)l,j}^{\parallel} X_{(1)l,k}^{\perp} \right], \quad (3.22)$$



$$\epsilon_{ijk} X_{(3c)k,j}^\perp = \frac{2}{3} \epsilon_{ijk} \left[ X_{(1)l,j}^\perp X_{(2)l,k}^\perp - 2X_{(1)l,j}^\parallel X_{(2b)k,l}^\parallel \right], \quad (3.23)$$

$$\epsilon_{ijk} X_{(3d)k,j}^\perp = \frac{1}{3} \epsilon_{ijk} X_{(1)l,j}^\perp X_{(2b)k,l}^\parallel. \quad (3.24)$$

Note that even though the irrotational condition is assumed, the non-zero transverse mode exists in the first term in the right-hand side of Eq. (3.20), as Buchert<sup>11)</sup> pointed out. In deriving (3.21)~(3.24), by using the following relation

$$\epsilon_{ijk} X_{(1)j,m}^\parallel \left[ X_{(1)m,l}^\parallel X_{(1)l,k}^\perp - X_{(1)k,l}^\parallel X_{(1)l,m}^\perp \right] = \frac{1}{3} \epsilon_{ijk} X_{(1)j,l}^\parallel \left[ X_{(2)l,k}^\perp - X_{(2)k,l}^\perp \right], \quad (3.25)$$

$$\epsilon_{ijk} X_{(1)j,l}^\parallel X_{(1)l,m}^\perp X_{(1)m,k}^\perp = \frac{1}{3} \epsilon_{ijk} X_{(1)l,j}^\parallel \left[ X_{(2)l,k}^\perp - X_{(2)k,l}^\perp \right], \quad (3.26)$$

we write the triplet of  $\mathbf{X}_{(1)}$  as the term including the second-order quantity. By the same reason in the case of the longitudinal solution, the term proportional to  $\ln t$  exists in the third-order transverse solution.

#### 3.4. Relation between vorticity and density fluctuation

In this subsection, we calculate the concrete form of the density fluctuation and the vorticity by using the Lagrangian perturbation formalism we have obtained above.

From the definition of the density fluctuation Eq. (2.1), we obtain the density fluctuation as follows:

$$\begin{aligned} \delta &= \frac{1}{\det [\delta_{ij} + A_{(1)ij} + A_{(2)ij} + A_{(3)ij}]} - 1 \\ &= \frac{1}{1 + J_{(1)} + J_{(2)} + J_{(3)} + \dots} - 1, \end{aligned} \quad (3.27)$$

where

$$J_{(1)} = t^{2/3} \nabla_q \cdot \mathbf{X}_{(1)}^\parallel, \quad (3.28)$$

$$J_{(2)} = \frac{2}{7} t^{4/3} \left[ \left( \nabla_q \cdot \mathbf{X}_{(1)}^\parallel \right)^2 - X_{(1)i,j}^\parallel X_{(1)j,i}^\parallel \right] - t^{1/3} X_{(1)i,j}^\parallel X_{(1)j,i}^\perp + \frac{1}{4} t^{-2/3} X_{(1)i,j}^\perp X_{(1)j,i}^\perp, \quad (3.29)$$

$$\begin{aligned} J_{(3)} &= t^2 \left[ \frac{4}{9} \left\{ X_{(1)i,i}^\parallel X_{(2a)j,j}^\parallel - X_{(1)i,j}^\parallel X_{(2a)j,i}^\parallel \right\} + \frac{2}{3} \det(X_{(1)i,j}^\parallel) \right] \\ &\quad + t \left[ -\frac{2}{3} X_{(1)i,j}^\parallel X_{(2)j,i}^\perp + \frac{4}{3} X_{(2a)i,j}^\parallel X_{(1)j,i}^\perp \right] \\ &\quad + \left[ -X_{(1)i,j}^\perp X_{(2)j,i}^\perp + \left\{ X_{(1)i,i}^\parallel X_{(2b)j,j}^\parallel - X_{(1)i,j}^\parallel X_{(2b)j,i}^\parallel \right\} \right. \\ &\quad \left. + X_{(1)i,j}^\parallel X_{(1)j,k}^\perp X_{(1)k,i}^\perp - \frac{1}{2} X_{(1)i,j}^\perp X_{(1)j,i}^\perp X_{(1)l,l}^\parallel \right] \\ &\quad + \frac{1}{75} t^{-1} (20 \ln t - 63) X_{(2b)i,j}^\parallel X_{(1)j,i}^\perp \\ &\quad + \frac{1}{75} t^{-1} (19 - 10 \ln t) X_{(1)i,j}^\perp X_{(1)j,k}^\perp X_{(1)k,i}^\perp. \end{aligned} \quad (3.30)$$

Although we represent  $J$  only up to the third-order here, we can calculate  $J$  up to the ninth-order. Recall that it is unnecessary that the density fluctuation is expanded perturbatively in spite of  $J$  being the perturbative quantity. By linearizing Eq. (3.27), we recover the Eulerian linear behavior presenting that the vorticity decouples from the density enhancement. From the quantities  $J_{(2)}$  and  $J_{(3)}$ , we see that the coupling of the rotational mode  $\mathbf{X}_{(1)}^\perp$  to the growing mode  $\mathbf{X}_{(1)}^\parallel$  plays an important role in the enhancement of the density fluctuation. Because this coupling is growing. The decaying mode we have omitted also can contribute to the enhancement of the density fluctuation. If we write the term containing one  $\mathbf{X}_{(1d)}^\parallel$  and two  $\mathbf{X}_{(1)}^\parallel$  in  $J_{(3)}$ , this term is proportional to  $t^{1/3}$ , namely,  $t^{-1}(t^{2/3})^2$ . However, since the term coupling of the rotational mode is proportional to  $t$ , the rotational mode makes more important contribution to the enhancement of the density fluctuation than the decaying mode. The importance of the coupling of the rotational mode to the growing mode is also seen from vorticity-density relation (see Eq. (2.12))

$$\mathbf{w} = \frac{1}{a^2} \frac{(\mathbf{w}(\mathbf{q}, t_0) \cdot \nabla_q)(\mathbf{q} + \mathbf{A}_{(1)} + \mathbf{A}_{(2)} + \mathbf{A}_{(3)})}{1 + J_{(1)} + J_{(2)} + J_{(3)} + \dots}. \quad (3.31)$$

The vorticity-density relation shows that the behavior of the vorticity depends on both the density fluctuation ( $\delta$  or  $J$ ) and the deformation of the fluid ( $x_{i,j}$  or  $A_{i,j}$ ). For realizing the concrete form of the vorticity, we shall expand Eq. (3.31) perturbatively. (Note that the vorticity is actually the non-perturbative quantity in the Lagrangian form.) We obtain

$$\mathbf{w}_{(1)} = \frac{1}{a^2} \mathbf{w}_{(1)}(\mathbf{q}), \quad (3.32)$$

$$\begin{aligned} \mathbf{w}_{(2)} = \frac{1}{a^2} \left[ t^{2/3} \left\{ (\mathbf{w}_{(1)}(\mathbf{q}) \cdot \nabla_q) \mathbf{X}_{(1)}^\parallel - \mathbf{w}_{(1)}(\mathbf{q}) (\nabla_q \cdot \mathbf{X}_{(1)}^\parallel) \right\} \right. \\ \left. + t^{-1/3} (\mathbf{w}_{(1)}(\mathbf{q}) \cdot \nabla_q) \mathbf{X}_{(1)}^\perp \right], \end{aligned} \quad (3.33)$$

$$\begin{aligned} \mathbf{w}_{(3)} = \frac{1}{a^2} \left[ t^{4/3} \left\{ \frac{1}{7} \mathbf{w}_{(1)}(\mathbf{q}) \left( 5 (\nabla_q \cdot \mathbf{X}_{(1)}^\parallel)^2 + 2 X_{(1)a,b}^\parallel X_{(1)b,a}^\parallel \right) \right. \right. \\ \left. - (\mathbf{w}_{(1)}(\mathbf{q}) \cdot \nabla_q) \mathbf{X}_{(1)}^\parallel (\nabla_q \cdot \mathbf{X}_{(1)}^\parallel) + (\mathbf{w}_{(1)}(\mathbf{q}) \cdot \nabla_q) \mathbf{X}_{(2a)}^\parallel \right\} \\ + t^{1/3} \left\{ \mathbf{w}_{(1)}(\mathbf{q}) X_{(1)a,b}^\parallel X_{(1)b,a}^\perp - (\mathbf{w}_{(1)}(\mathbf{q}) \cdot \nabla_q) \mathbf{X}_{(1)}^\perp (\nabla_q \cdot \mathbf{X}_{(1)}^\parallel) \right. \\ \left. + (\mathbf{w}_{(1)}(\mathbf{q}) \cdot \nabla_q) \mathbf{X}_{(2)}^\perp \right\} \\ \left. + t^{-2/3} \left\{ -\frac{1}{4} \mathbf{w}_{(1)}(\mathbf{q}) X_{(1)a,b}^\perp X_{(1)b,a}^\perp + (\mathbf{w}_{(1)}(\mathbf{q}) \cdot \nabla_q) \mathbf{X}_{(2b)}^\parallel \right\} \right]. \end{aligned} \quad (3.34)$$

Here, we assume

$$\mathbf{w}(\mathbf{q}, t_0) := \mathbf{w}_{(1)}(\mathbf{q}) = -\frac{1}{3} \nabla_q \times \mathbf{X}_{(1)}^\perp. \quad (3.35)$$

Same results are also derived from Eq. (2.11), that is, the transformation of the Eulerian coordinates to the Lagrangian one (see Appendix C). The results (3.32)

is the same as the behavior of the vorticity in Eulerian description. Furthermore, from the results (3.33) and (3.34), we see that the amplitude of induced vorticity becomes strong by the coupling of the rotational mode (initial vorticity) to many growing modes. This shows that the primordial vorticity is effective in the high-density regions. The exact calculation of the vorticity Eq. (3.31) will confirm this fact.

#### §4. Summary and conclusion

In this paper, we have investigated the relation between the vorticity and the density fluctuation by preparing the rotational Lagrangian perturbation theory. The fact that the vorticity couples to the density enhancement is already discussed by Buchert.<sup>4)</sup> However, we obtain more concrete vorticity-density relation by solving the perturbative equations up to the third-order for both longitudinal and transverse displacement vector. The coupling of the transverse mode to the growing mode enables the vorticity to contribute to the enhancement of the density fluctuation. Furthermore, growing vortical modes are induced by the coupling of initial vorticity to the growing mode. These results show the fact that the primordial vorticity is not negligible in high-density regions. Therefore, we must reconsider the importance of the primordial vorticity effect on the density fluctuation. Now we also have pointed out the effects of the decaying mode. It is possible that the first-order decaying mode contributes to the enhancement of the density fluctuation by coupling to the growing mode. However, since the transverse mode is proportional to  $t^{-1/3}$  and the decaying mode to  $t^{-1}$ , the former makes more important role in the enhancement of the density fluctuation.

The results we obtained in this paper give a powerful tool to treat the high-density regions and behavior of the vorticity. It is of interest to investigate quantitatively how the vorticity affects the non-linear evolution of the density fluctuations in the universe. It will be the subject of future investigation.

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#### Appendix A

##### —— Detail of Lagrangian equation for longitudinal mode ——

For the perturbative calculation, we give the detail of the second term of the right-hand side of the longitudinal equation (2.9a) as follows:

$$\begin{aligned} \nabla_x \cdot \left[ \left( \dot{\mathbf{A}} \cdot \nabla_x \right) \mathcal{D}(\mathbf{A}) - (\nabla_x \cdot \mathcal{D}(\mathbf{A})) \dot{\mathbf{A}} \right] \\ = \dot{A}_{i,j} \mathcal{D}(A_{j,i}) - \dot{A}_{i,i} \mathcal{D}(A_{j,j}) \end{aligned}$$

$$\begin{aligned}
& +B_{ij}\dot{A}_{j,k}\mathcal{D}(A_{k,i}) + B_{ij}\mathcal{D}(A_{j,k})\dot{A}_{k,i} - B_{ij}\dot{A}_{j,i}\mathcal{D}(A_{l,l}) - B_{ij}\mathcal{D}(A_{j,i})\dot{A}_{l,l} \\
& +B_{ij,k}\dot{A}_j\mathcal{D}(A_{k,i}) - B_{ij,k}\dot{A}_k\mathcal{D}(A_{j,i}) \\
& +B_{ij}\left[B_{kl}\dot{A}_l\mathcal{D}(A_{j,k}) - B_{kl}\mathcal{D}(A_{l,k})\dot{A}_j\right]_{,i}.
\end{aligned} \tag{A.1}$$

## Appendix B

### — Calculation of $B_{ij}$ —

When we solve the perturbative equations, the quantities  $B_{(1)ij}$  and  $B_{(2)ij}$  are necessary. So we present the derivation of their quantities here.

The quantity  $B_{ij}$  is defined by using the cofactor  $C_{ij}$  as follows:

$$B_{ij} = \sum_{n=1} \left( J^{-1} C_{ji} \right)_{(n)}, \quad C_{ji} = \frac{1}{2} \epsilon_{jab} \epsilon_{icd} x_{a,c} x_{b,d}. \tag{B.1}$$

Using Eq. (B.1), we obtain

$$B_{(1)ij} = -A_{(1)i,j}, \tag{B.2}$$

$$B_{(2)ij} = A_{(1)i,k} A_{(1)k,j} - A_{(2)i,j}. \tag{B.3}$$

## Appendix C

### — Useful relations to obtain the expression of the vorticity —

We have presented the expression of the vorticity in Eqs. (3.32), (3.33) and (3.34). These results are derived by vorticity-density relation Eq. (2.12). We also obtain the same results from Eq. (2.11), that is, the transformation of the Eulerian coordinates to the Lagrangian one. We list the useful relations to obtain the expression of the vorticity from Eq. (2.11) as follows:

$$\epsilon_{lmn} X_{(1)n,m}^\perp X_{(1)i,l}^\perp = \epsilon_{ijk} X_{(1)j,l}^\perp X_{(1)l,k}^\perp, \tag{C.1}$$

$$\epsilon_{lmn} X_{(1)n,m}^\perp X_{(1)i,l}^\parallel = \epsilon_{ijk} \left[ X_{(1)k,j}^\perp \left( \nabla_q \cdot \mathbf{X}_{(1)}^\parallel \right) + X_{(1)j,l}^\parallel \left( X_{(1)l,k}^\perp - X_{(1)k,l}^\perp \right) \right], \tag{C.2}$$

$$\epsilon_{ijk} X_{(1)k,j}^\perp X_{(1)l,m}^\perp X_{(1)m,l}^\perp = -2\epsilon_{ijk} X_{(1)j,l}^\perp X_{(1)l,m}^\perp X_{(1)m,k}^\perp, \tag{C.3}$$

$$\epsilon_{lmn} X_{(1)n,m}^\perp X_{(2)i,l}^\perp = \epsilon_{ijk} \left[ X_{(1)j,l}^\perp - X_{(1)l,j}^\perp \right] X_{(2)l,k}^\perp, \tag{C.4}$$

$$\begin{aligned}
& \epsilon_{ijk} X_{(1)l,j}^\perp \left[ X_{(2)l,k}^\perp - X_{(2)k,l}^\perp \right] \\
& = 3\epsilon_{ljk} X_{(1)l,m}^\parallel X_{(1)i,j}^\perp X_{(1)m,k}^\perp \\
& = 3\epsilon_{ijk} X_{(1)j,l}^\parallel X_{(1)l,m}^\perp X_{(1)m,k}^\perp \\
& = 3\epsilon_{ijk} \left[ X_{(1)j,m}^\parallel X_{(1)l,m}^\perp X_{(1)k,l}^\perp - X_{(1)k,j}^\parallel X_{(1)l,m}^\perp X_{(1)m,l}^\perp + X_{(1)l,m}^\parallel X_{(1)m,j}^\perp X_{(1)k,l}^\perp \right] \\
& \quad + 3\epsilon_{lmn} X_{(1)n,m}^\perp X_{(1)i,l}^\perp \left( \nabla_q \cdot \mathbf{X}_{(1)}^\parallel \right),
\end{aligned} \tag{C.5}$$

$$\begin{aligned}
& \epsilon_{ijk} X_{(1)j,l}^{\parallel} \left[ X_{(2)l,k}^{\perp} - X_{(1)k,l}^{\perp} \right] \\
&= 3\epsilon_{ijk} X_{(1)j,l}^{\parallel} \left[ X_{(1)l,m}^{\parallel} X_{(1)m,k}^{\perp} - X_{(1)k,m}^{\parallel} X_{(1)m,l}^{\perp} \right] \\
&= 3 \left[ \epsilon_{lmn} X_{(1)n,m}^{\perp} X_{(1)i,l}^{\parallel} \left( \nabla_q \cdot \mathbf{X}_{(1)}^{\parallel} \right) + \epsilon_{ijk} X_{(1)j,l}^{\parallel} X_{(1)l,m}^{\parallel} X_{(1)k,m}^{\perp} \right] \\
&\quad - \frac{3}{2} \epsilon_{ijk} X_{(1)k,j}^{\perp} \left[ \left( \nabla_q \cdot \mathbf{X}_{(1)}^{\parallel} \right)^2 + X_{(1)l,m}^{\parallel} X_{(1)m,l}^{\parallel} \right]. \tag{C.6}
\end{aligned}$$

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